AD-A192 714 APPROXIMATIONS OF STOCHASTIC EQUATIONS DRIVEN BY PREDICTABLE PROCESSES (U) BROWN UNIV PROVIDENCE RI LEFSCHETZ CENTER FOR DYNAMICAL SYSTE G FERREYRA 1/1

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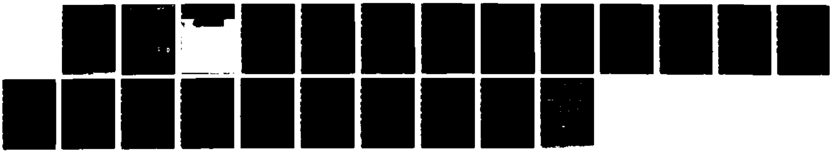

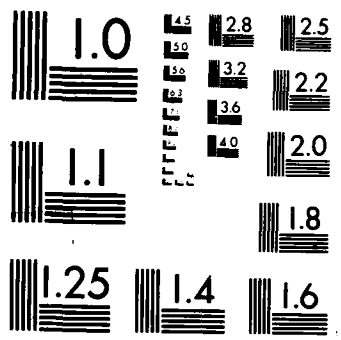
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by

Guillermo Ferreyra

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December 1987 LCDS/CCS #87-39

Approximation of Stochastic Equations

Driven by Predictable Processes

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Abstract

A theory of stochastic integral equations driven by predictable processes in Stratonovich sense is developed. These driving processes include a large class of discontinuous semimartingales. The theory of stochastic differential equations driven by continuous semimartingales in Stratonovich sense is extended without involving Lebesgue-Stieltjes integrals as done by Meyer. Moreover, a change of variables formula without extra terms involving the jumps of the processes holds for this theory. Results on approximation of driving processes are preserved.

Key words and phrases: stochastic equations, approximation of driving processes, Stratonovich integration.

§ 1. Introduction

We propose an interpretation for the expression

$$(1) \qquad dX(t) = f(X(t))\circ du(t) + g(X(t))\,dt + \sum_{\nu=1}^{N} \sigma_\nu(X(t)) \circ dW_\nu(t) ,$$

where u(t) is a uniformly bounded, adapted, left continuous or, more generally, predictable process, and $(W_1(t),\cdots,W_N(t))$ is an N-dimensional Brownian motion process. Our goal is to define what (1) means and to study the solutions of such equation. We extend here the theory of stochastic differential equations driven by continuous semimartingales in Stratonovich sense. In fact, if u(t) has Lipschitz paths with a uniform Lipschitz constant, we interpret (1) in the usual Stratonovich sense. The basic difficulty with (1) is that if the process u(t) is discontinuous at say $t = \tau$ (a predictable stopping time) then X(t) may also be discontinuous at $t = \tau$. Thus in general $\int_{\tau-\epsilon}^{\tau+\epsilon} f(X(s))du(s), \epsilon>0$, cannot be defined as a pathwise Riemann-Stieltjes integral. One way to get around this difficulty is to interpret this integral as a pathwise Lebesgue-Stieltjes integral. This approach is followed by Meyer [9]. But then the change of variables formula is burdened with terms that account for the jumps of X(t) (cf. Meyer [9], p. 301). These terms also complicate the equations that follow from using such generalized Ito formula in deducing evolution equations for various statistics of X(t). We interpret the first integral in (1) differently. This accounts for the notation $\circ du(t)$ which stands for an extension of Stratonovich integration. Our interpretation of (1) preserves two important properties. First, the usual change of variables formula holds for (1). That is, no extra terms due to the jumps of u(t) or X(t) appear in the formula. And second, robustness in u(t) is built into this interpretation. This is particularly important when u(t) is a control process. Simple cases of (1) are used to model the controlled state in problems of singular control (cf. Benes et al. [1], Karatzas and Shreve [7], Harrison [5], and Taksar [12]). In some applications, the controls u(t) that

appear naturally are continuous. This last property is nonrigorously dropped in order to find optimal-in some sense-controls (cf. Harrison [5], §5). The definitions we will introduce are oriented towards making rigorous that procedure. In fact, we will show that if $u(t)$ is approximated by Lipschitz processes $u_j(t)$ then the corresponding solutions $X_j(t)$ converge to $X(t)$ as $j \to \infty$. The sense in which these limits are taken is given later.

We mention a few papers dealing with problems related to ours. Kushner [8] contains an approximation theorem for jump-diffusion processes. The jump process is Poisson and weak convergence of approximations is proved. In our work we allow $u(t)$ to be a predictable process. Moreover, we prove that the approximations $X_j(t)$ converge strongly. Picard [10] studies approximations for stochastic differential equations driven by continuous martingales. Adapted and non adapted approximations of Brownian motion are considered. Strong convergence is obtained. Protter [11] allows general semimartingales (i.e. those with jumps) as differentials. Given decompositions of the semimartingales as the sum of three terms, a continuous semimartingale, a purely discontinuous local martingale and a process of bounded variation, the first two terms are approximated by smoother processes, but the bounded variation processes are left fixed. Thus our results are in some sense complementary to those of Protter [11]. Doleans-Dade [2] treats existence and uniqueness for stochastic integral equations with differentials of possibly discontinuous semimartingales. Lebesgue-Stieltjes integrals are used in [2] when differentials of bounded variation processes appear. The relation between our approach and that of using Lebesgue-Stieltjes integrals is further explored in Ferreyra [3]. To clarify ideas we begin with a simple example in §2. Section 3 contains our hypotheses. Existence and iniqueness for (1) is treated in §4. The chain rule is considered in §5.

§2. An Example.

It is enough to consider deterministic functions to illustrate the basic ideas.

Consider the problem

$$dX(t) = X(t)\ du(t), \quad 0 \leq t \leq 2,$$
$$X(0) = 1,$$

where $u(t) = 1_{(1,2)}(t)$, $0 \leq t \leq 2$, is the characteristic function of the interval (1,2) defined for $t \in [0,2]$. If $I(t) = \int_0^t X(s)\ du(s)$ is interpreted as a Lebesgue-Stieltjes integral, then $I(t) = X(1)\ 1_{(1,2)}(t)$. Thus the above initial value problem is solved by

$$X(t) = 1 + I(t) = 1 + 1_{(1,2)}(t).$$

This solution is well defined for all $t \in [0,2]$, and it jumps one unit at $t = 1$. The following change of variables formula holds for X(t) and suitable φ (cf. Meyer [9], p. 301).

$$\varphi(X(t)) = \varphi(X(0)) + \int_0^t \varphi'(X(s))\ dX(s) + \{\varphi(X(1^+)) - \varphi(X(1))$$
$$- \varphi'(X(1))\ (X(1^+) - X(1))\}\ 1_{(1,2]}(t).$$

Next, consider the expression

$$d\bar{X}(t) = \bar{X}(t) \circ du(t), \ 0 \leq t \leq 2, \quad \bar{X}(0) = 1 .$$

Our approach for interpreting and solving this is as follows. Approximate u(t) by

$$u_j(t) = j(t-1)\ 1_{(1,1+1/j\,]}(t) + 1_{(1+1/j,2]}(t).$$

Now, solve

$$dX_j(t) = X_j(t)\ du_j(t), \quad X_j(0) = 1.$$

The solution $X_j(t)$ equals 1 for $t \leq 1$, exp $[j\,(t-1)]$ for $1 \leq t \leq 1 + 1/j$, and e for $t > 1 + 1/j$. Finally, define $\bar{X}(t) = \lim_{j \to \infty} X_j(t)$. Then $\bar{X}(t)$ equals 1 for $t \leq 1$, and e for $t > 1$. It is easy to check now that

$$d\varphi(\bar{X}(t)) = \varphi'(\bar{X}(t)) \circ du(t), \quad \varphi(\bar{X}(0)) = \varphi(1).$$

In fact, properties of the Riemann integral give

$$d\varphi(X_j(t)) = \varphi'(X_j(t))\ du_j(t), \ \varphi(X_j(0)) = \varphi(1),$$

and $\varphi(\bar{X}(t)) = \lim_{j \to \infty} \varphi(X_j(t))$.

§3. Notation and hypotheses.

Let (Ω,F,P) be a complete probability space and let (F_t), $0\le t\le T$, be a right continuous increasing family of sub σ-fields of F each containing all P-null sets. Let $(W_1(t),\cdots,W_N(t))$, $0\le t\le T$, be an N-dimensional (F_t)-Brownian motion. We are concerned with the expression

$$(1) \qquad dX(t) = f(X(t)) \circ du(t) + g(X(t))\, dt + \sum_{\nu=1}^{N} \sigma_\nu(X(t)) \circ dW_\nu(t) .$$

The small circle $\circ$ in the first term on the right hand side of (1) is introduced since that term does not stand for a Lebesgue-Stieltjes type integral. In fact, this integration is an extension of Stratonovich integration.

Let V be the set of real valued processes $v(t)$, $0\le t\le T$, that are adapted, uniformly bounded (in t and ω), and continuous on the left. Let U be the set of real valued processes $v(t)$, $0\le t\le T$, that are uniformly bounded and predictable. (Predictable means that $(t,\omega)\to u(t,\omega)$ is measurable with respect to the σ-field on $[0,T]\times\Omega$ generated by the left-continuous, (F_t)-adapted processes). Assume, without loss of generality, that all processes in V and U satisfy $v(0) = 0$. In section 4 we assume that $u\in V$, while in section 5 we assume $u\in U$. We assume throughout that $u(t) = 0$ for $t < 0$. The unknown process $X(t)$, $0\le t\le T$, evolves in $\mathbb{R}^n$. The initial data $X(0) = X$ is an F_0-measurable, $\mathbb{R}^n$-valued random vector such that

(H1) $E|x|^p<\infty$, for some $p>2$.

Finally, the coefficients f, g and σ_ν, $\nu = 1,\cdots,N$, are vector fields on $\mathbb{R}^n$ such that

(H2) $f\in C^3(\mathbb{R}^n)$, $f_x\in C_b^2(\mathbb{R}^n)$,

(H3) $g\in C_b^1(\mathbb{R}^n)$, and

(H4) $\sigma_\nu\in C_b^2(\mathbb{R}^n)$, $\nu = 1,\cdots,N$.

Here $C_b^i(\mathbb{R}^n)$, $i = 1, 2$, is the subset of functions in $C^i(\mathbb{R}^n)$ which are bounded together with all their partial derivatives up to order i.

Definition 1: We say that a real-valued process $v(t)$, $0 \leq t \leq T$, belongs to $\mathfrak{X}$ if it is (F_t)-adapted, and it has Lipschitz paths with a uniform Lipschitz constant.

It is well known that under our hypotheses on the coefficients and assuming $v \in \mathfrak{X}$, the equation

$$(2) \qquad dX(t) = \left[f(X(t)) \frac{dv}{dt}(t) + g(X(t))\right]dt + \sum_{\nu=1}^{N} \sigma_\nu(X(t)) \circ dW_\nu(t),$$

has a unique solution given that $X(0) = X$.

Next, we consider the approximation of the process u by processes in $\mathfrak{X}$.

Lemma 1: (a) Let $v \in V$. Then there exists a uniformly bounded sequence $\{v_j\}$ of elements of $\mathfrak{X}$ such that for each $0 \leq t \leq T$, $v_j(t) \to v(t)$, a.e.

(b) Let $v \in U$, and let $1 \leq p' < \infty$. Then there exists a uniformly bounded sequence $\{v_j\}$ of elements of $\mathfrak{X}$ such that

$$E\int_0^T |v_j(t) - v(t)|^{p'} dt \to 0, \text{ a.s. as } j \to \infty.$$

Proof: (a) Given $v \in V$, define $v_j(t) = j\int_{t-1/j}^{t} v(s)ds$, $j = 1, 2, \ldots$. Here we assume $v(t) = 0$ for $t < 0$. Then $v_j \in \mathfrak{X}$ since v is uniformly bounded and adapted. Moreover, since v is left continuous, we have for each t, $0 \leq t \leq T$, $v_j(t) \to v(t)$, a.s.

To prove the second part of the lemma, we use the following result.

Lemma 2: (Ikeda-Watanabe [6], p. 21): Let Φ be a linear space of real measurable processes which are uniformly bounded (as functions from $[0,T] \times \Omega$ into $\mathbb{R}$). Assume Φ satisfies the following conditions.

(i) Φ contains all uniformly bounded, left continuous, (F_t)-adapted processes; i.e., $V \subset \Phi$. and

(ii) if $\{\phi_n\}$ is a monotone increasing sequence of processes in Φ such that $\phi = \sup \phi_n$ is uniformly bounded, then $\phi \in \Phi$.

Then Φ contains all uniformly bounded predictable processes; i.e., $U \subset \Phi$.

Proof of part (b) of Lemma 1: Let Φ be the set of $v \in U$ such that the conclusion of part (b) holds. Clearly Φ is a linear space. To prove that Φ satisfies the condition (i) of Lemma 2, let $v \in V$. Then the sequence $\{v_j\}$ defined as in the proof of part (a) of Lemma 1 satisfies

$$\lim_{j\to\infty} E\int_0^T |v_j(t) - v(t)|^{p'} dt = 0$$

by the Bounded Convergence Theorem. Then $V \subset \Phi$. To prove condition (ii) let $\{\phi_j^n\}$ be a sequence approximating ϕ_n in the sense of (b). Also, by the Bounded Convergence Theorem

$$\lim_{j\to\infty} E\int_0^T |\phi_n(t) - \phi(t)|^{p'} dt = 0.$$

Then approximate $\phi = \sup \phi_n$ as follows. Given $j \in \{1, 2, \ldots\}$ choose n such that $E\int_0^T |\phi_n(t) - \phi(t)|^{p'} dt < \frac{1}{2j}$. Next, choose λ such that $E\int_0^T |\phi_m^n(t) - \phi_n(t)|^{p'} dt < \frac{1}{2j}$ for all $m \geq \lambda$. Finally, let $\Psi_j = \phi_\lambda^n$. Then $\{\Psi_j\}$ is a uniformly bounded sequence of elements of $\mathfrak{X}$ satisfying

$$\lim_{j\to\infty} E\int_0^T |\Psi_j(t) - \phi(t)|^{p'} dt = 0, \text{ a.s.}$$

Thus $\phi \in \Phi$ and the proof of Lemma 1 is concluded.

§4. Existence and uniqueness when u is in V.

Definition 2: Assume $u \in V$ is given. Then, an $\mathbb{R}^n$-valued process $X(t)$, $0 \leq t \leq T$, is said to be a solution of (1) with initial condition $X(0) = X$ if there exists a map $\Gamma: [0,T] \times V \times \Omega \to \mathbb{R}^n$ such that the following conditions are satisfied (the dependence of Γ on $\omega \in \Omega$ is not displayed below).

(D.i) For all $v \in V$, $\Gamma(t,v)$ is $(\mathcal{F}_t)$-adapted.

(D.ii) For all $v \in V$, $\Gamma(0,v) = X$.

(D.iii) If $v \in \mathfrak{X}$, then the process $\Gamma(t,v)$ solves (2) in Stratonovich sense.

(D.iv) If $v \in V$ and $\{v_j\}$ is a uniformly bounded sequence of elements in V such that for

every t, $0 \le t \le T$, $v_j(t) \to v(t)$, a.s., then for each t, $0 \le t \le T$,

$E|\Gamma(t,v_j) - \Gamma(t,v)|^2 \to 0$, $j \to \infty$.

(D.v) $\Gamma(t,u) = X(t)$, $0 \le t \le T$.

Remark 1: If v and $\{v_j\}$ are as in (D.iv) then the Bounded Convergence Theorem implies that for every $1 \le p' < \infty$, $\int_0^T E|v_j(s) - v(s)|^{p'} ds \to 0$, as $j \to \infty$.

Theorem 1: Assume that conditions (H1) through (H4) hold. Then, given $u \in V$, the system (1) with initial condition $X(0) = X$ has a solution $X(t)$, $0 \le t \le T$, in the sense of Definition 2 such that

(a) for each t, X(t) is unique in quadratic-mean norm, and

(b) $E|X(t)|^p$ is bounded, uniformly in t.

Proof:

(a) Uniqueness: Suppose X(t) and $\bar{X}(t)$ are two competing solutions of (1) starting at X. Then, there exist two maps Γ, $\bar{\Gamma}$: $[0,T] \times V \times \Omega \to \mathbb{R}^n$ satisfying $\Gamma(t,u) = X(t)$, $\bar{\Gamma}(t,u) = \bar{X}(t)$ and properties (D.i) through (D.iv). Consider a uniformly bounded sequence $\{u_j\}$ of elements of $\mathfrak{X}$ such that for each t, $0 \le t \le T$, $u_j(t) \to u(t)$, a.s. The existence of $\{u_j\}$ follows from part (a) of Lemma 1. Then, for each j, the processes $\Gamma(t,u_j)$ and $\bar{\Gamma}(t,u_j)$ are solutions of the same equation (2) with identical initial condition. But uniqueness holds in this case since $u_j \in \mathfrak{X}$. Then $\bar{\Gamma}(t,u_j) = \Gamma(t,u_j)$. Then (D.iv) imply that for each t, $0 \le t \le T$, $E|X(t) - \bar{X}(t)|^2 = 0$. Thus (a) is proved.

Existence: As a first step we reduce the problem of solving (1) to that of solving a simpler system. For this, we introduce a transformation which was also used in Sussmann [13] for a purpose similar to ours. Let F: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow of f, that is, the solution of

$$\frac{\partial F}{\partial s}(s,x) = f(F(s,x)) \ , \ (s,x) \in \mathbb{R} \times \mathbb{R}^n \ , \ F(0,x) = x \ .$$

Then $F \in C^3(\mathbb{R} \times \mathbb{R}^n)$ and if $A(s,x)$ denotes the $n \times n$ matrix $\frac{\partial F}{\partial x}(s,x)$, then

$$(3) \qquad A(s,x) = I + \int_0^s \frac{\partial f}{\partial x}(F(\tau,x))\, A(\tau,x)\, d\tau ,$$

where I is the $n \times n$ identity matrix. Let $B(s,x)$ denote the inverse of $A(s,x)$. Then

$$(4) \qquad B(s,x) = I - \int_0^s B(\tau,x) \frac{\partial f}{\partial x}(F(\tau,x))\, d\tau .$$

Since $\frac{\partial f}{\partial x}$ is bounded, then it follows from Gronwall's inequality that both $A(s,x)$ and $B(s,x)$ are bounded as long as s remains bounded. Also, it is easy to deduce from the above formulae and (H2) that

$$(5) \qquad |F(s,x)| + |F_s(s,x)| \le \beta(s)\ (1 + |x|) ,$$

where β is independent of x and it is bounded as long as s remains bounded.

Introduce the following vector fields on $\mathbb{R}^{n+1}$. Let

$$\bar{f}(s,x) = (1,0,\cdots,0) ,$$

$$\bar{g}(s,x) = (0,\ B(s,x)\ g(F(s,x))) , \text{ and}$$

$$\bar{\sigma}_\nu(s,x) = (0,\ B(s,x)\ \sigma_\nu(F(s,x))) , \quad \nu = 1,\cdots, N .$$

Let $F_*(s,x)$ be the Jacobian matrix of F at (s,x). Then

$$(6) \qquad F_*(s,x)\ \bar{f}(s,x) = F_s(s,x) = f(F(s,x)) ,$$

$$(7) \qquad F_*(s,x)\ \bar{g}(s,x) = g(F(s,x)) ,$$

$$(8) \qquad F_*(s,x)\ \bar{\sigma}_\nu(s,x) = \sigma_\nu(F(s,x)) .$$

The following lemma is needed to solve a stochastic differential equation in Stratonovich sense involving $\bar{g}$ and $\bar{\sigma}_\nu$, $\nu = 1,\cdots N$, as coefficients.

Lemma 3: *Let $\bar{\sigma}$ denote either one of the vector fields $\bar{\sigma}_\nu$, $\nu = 1,\cdots N$. We have $\bar{\sigma} \in C^2(\mathbb{R} \times \mathbb{R}^n)$ and $\bar{g} \in C^1(\mathbb{R} \times \mathbb{R}^n)$. Moreover, $\bar{\sigma}$, $\frac{\partial \bar{\sigma}}{\partial x_i}$, $\frac{\partial^2 \bar{\sigma}}{\partial x_i \partial x_j}$, $\bar{g}$ and $\frac{\partial \bar{g}}{\partial x_i}$, $i,j = 1,\cdots,n$, are bounded as functions of (s,x) as long as s remains in a bounded set. On the other hand,*

$$\left| \frac{\partial \bar{\sigma}}{\partial s}(s,x) \right| + \sum_{i=1}^{n} \left| \frac{\partial^2 \bar{\sigma}}{\partial s \partial x_i}(s,x) \right| + \left| \frac{\partial \bar{g}}{\partial s}(s,x) \right| \le \beta(s)\,(1 + |x|),$$

where $\beta(s)$ is bounded as long as s remains in a bounded set.

The proof of these properties follows easily from repeated use of (3) - (5) , (H2) - (H4) and Gronwall's inequality.

Consider now the system of $n + 1$ equations

$$d\bar{Y}(t) = \bar{f}(\bar{Y}(t)) \circ dv(t) + \bar{g}(\bar{Y}(t))\,dt + \sum_{\nu=1}^{N} \sigma_\nu(\bar{Y}(t)) \circ dW_\nu(t),$$

where $\bar{Y}(t)$ is $\mathbb{R}^{n+1}$ - valued, and $v \in \mathfrak{X}$. Then, this is equivalent to

$$\text{(9)} \qquad dY(t) = \tilde{g}(v(t), Y(t))\,dt + \sum_{\nu=1}^{N} \tilde{\sigma}_\nu(v(t), Y(t)) \circ d W_\nu(t),$$

where $Y(t)$ is $\mathbb{R}^n$-valued, and $\tilde{g}$ (resp. $\tilde{\sigma}_\nu$) stands for the last n components of $\bar{g}$ (resp. $\bar{\sigma}_\nu$), i.e., $\tilde{g} = B(g \circ F)$ (resp. $\tilde{\sigma}_\nu = B(\sigma_\nu \circ F)$) . The system with Itô differentials equivalent to (9) is

$$\text{(10)} \qquad dY(t) = \tilde{h}(v(t), Y(t))\,dt + \sum_{\nu=1}^{N} \tilde{\sigma}_\nu(v(t), Y(t))\,dW_\nu(t),$$

where $\tilde{h} = \tilde{g} + \frac{1}{2} \sum_{\nu=1}^{N} \sum_{i=1}^{n} \frac{\partial \tilde{\sigma}_\nu}{\partial x_i} \tilde{\sigma}_\nu^i$, and $\tilde{\sigma}_\nu^i$ is the i-th component of $\tilde{\sigma}_\nu$.

The process $v(t)$ enters (10) as a parameter. It is well known that if $v(t)$ is (F_t) - adapted

and uniformly bounded then Lemma 3 implies that given the initial condition $Y(0) = X$, there exists a solution $Y(t)$ of (10) which is stochastically unique (see, for example, Gihman-Skorohod [4], pp. 50-52). Next, we prove that the p moment of $Y(t)$, where p is the power in (H1), is bounded uniformly in t. From (10) and the inequality $|a + b|^p \leq 2^p \{|a|^p + |b|^p\}$, it follows that

$$E\,|Y(t)|^p \leq C_1(T,p)\,\{E|\,X|^p + \int_0^t E|\,\tilde{h}(v(s),\,Y(s))|^p\,ds$$

$$+ \sum_{\nu=1}^{N} E\,|\int_0^t \tilde{\sigma}_\nu\,(v(s)\,,\,Y(s))\,dW_\nu\,(s)|^p\,\}$$

$$\leq C_2(T,p)\,\{\,E|\,X\,|^p + \int_0^t E\,|\,\tilde{h}\,(v(s),\,Y(s))\,|^p\,ds$$

$$+ \sum_{\nu=1}^{N} \int_0^t E\,|\tilde{\sigma}_\nu\,(v(s)\,,\,Y(s))\,|^p\,ds\},$$

where the last inequality is a consequence of the estimates for moments of stochastic integrals (see, for example, Zakai [15], p. 173). Now by Lemma 3, the coefficients $\tilde{h}$ and $\tilde{\sigma}_\nu$ of (10) are bounded provided $v(t)$ is uniformly bounded. Then, if M is a uniform bound for $v(t)$, it follows from (H1) that

(11) $\qquad E\,|Y(t)\,|^p \leq C_3(T,p,M\,) < \infty$, uniformly for $t \in [0,T]$.

Suppose next that $v \in V$ and $\{v_j\}$ is a uniformly bounded sequence of elements in V such that for every t, $0 \leq t \leq T$, $v_j(t) \to v(t)$, a.s. Let $Y_j\,(t)$ denote the solution of

$$dY_j\,(t) = \tilde{h}(v_j\,(t)\,,\,Y_j(t))\,dt + \sum_{\nu=1}^{N} \tilde{\sigma}_\nu(v_j(t)\,,\,Y_j(t))\,dW_\nu\,(t) \quad ,$$

with initial condition $Y_j\,(0) = X$. This and (10) imply

$$E|Y_j(t)-Y(t)|^2 \le C_4\ (T)\{\int_0^t E|\tilde{h}(v_j(s), Y_j(s)) - \tilde{h}(v_j(s), Y(s))|^2 ds$$

$$+\int_0^t E\ |\tilde{h}(v_j(s), Y(s)) - \tilde{h}(v(s), Y(s))|^2 ds$$

$$+\sum_{\nu=1}^{N}\int_0^t E\ |\tilde{\sigma}_\nu\ (v_j(s)\ , Y_j(s)) - \tilde{\sigma}_\nu\ (v_j(s), Y(s))|^2 ds$$

$$+\sum_{\nu=1}^{N}\int_0^t E\ |\tilde{\sigma}_\nu\ (v_j(s), Y(s)) - \tilde{\sigma}_\nu\ (v(s), Y(s))|^2 ds\}.$$

Now, The Mean Value Theorem together with the estimates for the derivatives of $\tilde{h}$ and $\tilde{\sigma}_\nu$ given by Lemma 3 imply

$$E|Y_j(t)-Y(t)\ |^2 \le C_5\ (T)\{\int_0^t E\ |Y_j\ (s) - Y(s)\ |^2\ ds$$

$$+\int_0^t E\,[(1+|\ Y(s)|^2)\ |\ v_j\ (s) - v(s)|^2\ ]\ ds\}.$$

Hölder's inequality together with (11) and Remark 1 imply that the last integral converges uniformly to zero as $j \to \infty$. Then Gronwall's inequality implies that

(12) $E|Y_j(t) - Y(t)|^2 \to 0$, as $j\to\infty$, uniformly for $t\in[0,T]$.

Define Γ: $[0,T] \times V \times \Omega \to \mathbb{R}^n$ as follows. Let

(13) $\Gamma(t,v) = F(v(t), Y(t))$,

where Y(t) is the solution of (10) with initial condition $Y(0) = X$ ($\omega\in\Omega$ is not shown) It is clear that Γ satisfies (D.i) and (D.ii). To check (D.iii) let $v\in\mathfrak{X}$. By the chain rule for Stratonovich integrals (cf. Ikeda-Watanabe [5], p. 101) and (6) - (8)

$$d\Gamma(t,v) = F_s(v(t), Y(t))dv(t) + F_x(v(t), Y(t)) \circ dY(t)$$

$$= f(F(v(t), Y(t)))\, dv(t) + g(F(v(t), Y(t)))\, dt + \sum_{\nu=1}^{N} \sigma_\nu(F(v(t), Y(t))) \circ dW_\nu(t)$$

$$= f(\Gamma(t,v))\, dv(t) + g(\Gamma(t,v))\, dt + \sum_{\nu=1}^{N} \sigma_\nu(\Gamma(t,v)) \circ dW_\nu(t) \quad .$$

Thus to conclude the proof of existence it remains to prove (D.iv). For this, let $\{v_j\}$ and v be as required in (D.iv). The Mean Value Theorem together with (13) imply

(14) $$E\,|\Gamma(t,v_j) - \Gamma(t,v)|^2 = E|\,F(v_j(t), Y_j(t)) - F(v(t), Y(t))|^2$$

$$\le C_6\, E\{\,|Y_j(t) - Y(t)\,|^2 + (1 + |Y(t)\,|^2)|v_j(t) - v(t)\,|^2\} \,.$$

Hölder's inequality, together with (11) and (12) imply that for each t, $0 \le t \le T$

(15) $$E\,|\Gamma(t,v_j) - \Gamma(t,v)\,|^2 \to 0 \quad , \text{ as } j \to \infty .$$

It remains to prove (b) to conclude the proof of Theorem 1. We have

(16) $$E\,|\Gamma(t,v)|^p = E\,|\,F(v(t), Y(t))|^p \le C_7(p)\,(1 + E\,|Y(t)|^p) \quad .$$

Then (11) implies (b).

Corollary: (a) For the solution X(t) of (1) to have a discontinuity at τ it is necessary (but not sufficient) that u(t) have a discontinuity at τ.

(b) If u(t) is continuous on the left, with limits on the right, so is X(t).

Proof: The solution Y(t) of (10) with Y(0) = X is a process with continuous paths. Then the jumps of the process $X(t) = \Gamma(t,u) = F(u(t),Y(t))$, with Γ and F as in Theorem 1, can only be produced by jumps of u(t). Hence our statements follow.

§ 5. Existence and uniqueness when u is in U.

Definition 3: Assume u in U is given. Then, an $\mathbb{R}^n$-valued process X(t), $0 \le t \le T$, is said to be a solution of (1) with initial condition X(0) = X if there exists a map $\Lambda: [0,T] \times U \times \Omega \to \mathbb{R}^n$ such that the following conditions are satisfied.

(E.i) For all $v \in U$, $\Lambda(t,v)$ is predictable.

(E.ii) For all $v \in U$, $\Lambda(0,v) = X$.

(E.iii) If $v \in \mathfrak{X}$, then the process $\Lambda(t,v)$ solves (2) in Stratonovich sense.

(E.iv) Let $q = 2 + 4/(p - 2)$; i.e., such that $2/p + 2/q = 1$. If $v \in U$ and $\{v_j\}$ is a uniformly bounded sequence of elements in U such that $\lim_{j\to\infty} E\int_0^T |v_j(t) - v(t)|^q dt = 0$, then $\lim_{j\to\infty} E\int_0^T |\Lambda(t,v_j) - \Lambda(t,v)|^2 dt = 0$.

(E.v) $\Lambda(t,u) = X(t)$, $0 \leq t \leq T$.

Theorem 2: Assume that conditions (H1) through (H4) hold. Then, given $u \in U$, the system (1) with initial condition $X(0) = X$ has a solution $X(t)$, $0 \leq t \leq T$, in the sense of Definition 3 such that

(a) $X(t)$ is unique in the norm of $L^2([0,T] \times \Omega)$, and

(b) $E\int_0^T |X(t)|^p \, dt < \infty$.

Proof: (a) Uniqueness: The proof is similar to that of uniqueness in Theorem 1. In fact, suppose Λ, $\bar{\Lambda}:[0,T] \times U \times \Omega \to R^n$ are two competing functions satisfying properties (E.i) through (E.iv). By part (b) of Lemma 1 let $\{u_j\}$ be a uniformly bounded sequence of elements in $\mathfrak{X}$ such that $E\int_0^T |u_j(t) - u(t)|^q dt \to 0$, as $j \to \infty$, with q as in (E.iv). Then $\Lambda(t,u_j)$ and $\bar{\Lambda}(t,u_j)$ are identical because uniqueness holds for (2). Then (E.iv) implies that $E\int_0^T |\Lambda(t,u) - \bar{\Lambda}(t,u)|^2 \, dt = 0$. Thus (a) is proved.

Existence: The proof of this part is the same as that of existence for Theorem 1 with a few minor changes. In the proof of (12) we use the hypothesis $E\int_0^T |v_j(t) - v(t)|^q \, dt \to \infty$, $j \to \infty$, of (E.iv) instead of using Remark 1. The function Λ is defined on $[0,T] \times U \times \Omega$ as $\Lambda(t,v) = F(v(t)),Y(t))$. Then the proof of (E.iv) follows from

$$E\int_0^T |\Lambda(t,v_j) - \Lambda(t,v)|^2 \, dt = E\int_0^T |F(v_j(t),Y_j(t)) - F(v(t),Y(t))|^2 \, dt$$

$$\leq C \, E\int_0^T \Big\{ |Y_j(t) - Y(t)|^2 + (1 + |Y(t)|^2) \, |v_j(t) - v(t)|^2 \Big\} dt,$$

and Hölder's inequality.

Finally, (b) follows from

$$E\int_0^T |\Lambda(t,v)|^p \, dt = E\int_0^T |F(v(t),Y(t))|^p \, dt \leq C' E\int_0^T \left\{1 + |Y(t)|^p\right\} dt.$$

§ 6. The chain rule.

We consider only the case of $u \in V$ in this section. Clearly, the chain rule can also be stated and proved with trivial modifications for the case of $u \in U$.

In this section we assume the hypotheses and notation of Theorem 1. Coordinates of vector valued functions are indicated by an upper index. For example, f^i denotes the i-th coordinate of f. We introduce the following extension of Definition 2.

Definition 4: Let $\psi \in C^3(\mathbb{R}^n)$. An $\mathbb{R}^n$ - valued process X(t), $0 \leq t \leq T$, is said to satisfy

(17) $$d\psi(X(t)) = \sum_{i=1}^{n} \frac{\partial\psi}{\partial x_i}(X(t)) \{f^i(X(t)) \circ du(t) + g^i(X(t))\, dt + \sum_{\nu=1}^{N} \sigma_\nu^i(X(t)) \circ dW_\nu(t)\},$$

(18) $$\psi(X(0)) = \psi(X),$$

if there exists a map Γ: $[0,T] \times V \times \Omega \to \mathbb{R}^n$ such that conditions (D.i), (D.ii), (D.iv), (D.v) and (D.vi) are satisfied.

Condition (D.vi) is as follows.

(D.vi) If $v \in \mathfrak{X}$, then $\Gamma(t,v)$, $0 \leq t \leq T$, satisfies

(19) $$d\psi(\Gamma(t,v)) = \sum_{i=1}^{N} \frac{\partial\psi}{\partial x_i}(\Gamma(t,v)) \{f^i(\Gamma(t,v)) \frac{dv}{dt}(t)\, dt + g^i(\Gamma(t,v))\, dt + \sum_{\nu=1}^{N} \sigma_\nu^i(\Gamma(t,v)) \circ dW_\nu(t)\},$$

(20) $$\psi(\Gamma(0,X)) = \psi(X),$$

in Stratonovich sense.

Remark 2: If $\psi(x) = x$, this is Definition 2.

Theorem 3: *Let* $\psi \in C^3(\mathbb{R}^n)$ *and let* $X(t)$, $0 \le t \le T$, *satisfy* (1) *with* $X(0) = X$, *in the sense of Definition* 2. *Then* $X(t)$, $0 \le t \le T$, *satisfies* (17) - (18).

Proof: For $v \in \mathfrak{X}$ the usual chain rule holds for $X(t)$ (cf. Ikeda-Watanabe [6], p. 101). That is, (D.vi) is satisfied. Then $X(t)$ satisfies (17) - (18) in the sense of Definition 4.

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